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Relativistic Geometrical Optics

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We investigate the gravitational and electromagnetic fields on the generalized Lagrange space endowed with the metric $g_{ij}(x, y)=\gamma_{ij}(x)+\{1+1/n^2(x, y)\}y_iy_j$. The generalized Lagrange spaces M^m do not reduce to Lagrange spaces. Consequently, they cannot be studied by methods of symplectic geometry. The restriction of the spaces M^m to a section $S_v(M)$ leads to the Maxwell equations and Einstein equations for the electromagnetic and gravitational fields in dispersive media with the refractive index $n(x, V)$ endowed with the Synge metric. When $n(x, V)=1$ we have the classical Einstein equations. If $1/n^2=1-1/c^2$ (c being the light velocity), we get results given previously by the authors. The present paper is a detailed version of a work in preparation.

INTRODUCTION

In two recent papers (Kawaguchi and Miron, 1989*a*,*b*), we studied some geometrical models for gravitation and electromagnetism considering the generalized Lagrange spaces $M^m=(M, g_{ij}(x, y))$ in which M is an m-dimensional manifold and $g_{ij}(x, y)$ is the metric tensor

$$g_{ij}(x, y)=\gamma_{ij}(x)+\frac{1}{c^2}y_iy_j, \qquad y_i=\gamma_{ij}(x)y^j \tag{a}$$

y^i is the Liouville vector field on the total space TM of the tangent bundle (TM, π, M), and $\gamma_{ij}(x)$ is a Riemannian metric on M.

Assuming that on M there exists a C^∞-vector field $V^i(x)$, $x\in M$, we can consider the cross section $S_V\colon M\to TM$ of the projection $\pi\colon TM\to M$, given by

$$x^i=x^i, \qquad y^i=V^i(x), \qquad x\in M \qquad (i=1, \ldots, m)$$

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Then the restriction of the metric (a) to the cross section $S_V(M)$ leads to the metric

$$g_{ij}(x, V(x)) = \gamma_{ij}(x) + \frac{1}{c^2} V_i V_j \tag{a'}$$

The metric was studied by Beil (1987, 1989) and used in some problems from electrodynamics. It is related to a new class of Finsler metric (Beil, 1989).

Therefore, our works (Kawaguchi and Miron, 1989*a,b*; Miron and Kawaguchi, 1991*a,b*) give geometrical models for gravitational and electromagnetic fields based on the metric (a).

R. G. Beil (private communication, October 1989) has pointed out the more general metric

$$g_{ij}(x, V(x)) = \gamma_{ij}(x) + \left(1 - \frac{1}{n^2(x, y)}\right) V_i V_j \tag{b}$$

where $n(x, V(x))$, $x \in M$, is the index of refraction of the medium.

The metric (a') appears as a particular case of the metric (b): $1/n^2 = 1 - 1/c^2$. This metric is discussed extensively by Synge (1968, pp. 376, 384), where its application to the propagation of electromagnetic waves in a medium with the index of refraction n is established.

Remarking that the metric (b) is the restriction to the cross section $S_V(M)$ of the d-tensor field

$$g_{ij}(x, y) = \gamma_{ij}(x) + \left(1 - \frac{1}{n^2(x, y)}\right) y_i y_j \tag{I}$$

we will study in the present paper the generalized Lagrange space $M^m = (M, g_{ij}(x, y))$ with the fundamental tensor (I) (which explains the title of this paper).

We prove that M^m is not reduced to a Finslerian or a Lagrangian space (Miron and Anastasiei, 1987). It is a generalized Lagrange space, a notation studied by Miron (1985) and extensively presented in Miron and Anastasiei (1987). The generalized Lagrange spaces were also studied by Aringazin and Asanov (1985), Asanov (1985), Aikou and Hashiguchi (1984), Anastasiei (1981), Atanasiu (1984), Hashiguchi (1984), Ichiyo (1988), Izumi (1987), Kawaguchi and Miron (1989*a,b*), Kikuchi (1988), Klepp (1982), Opris (1980), Rund (1982), Sakaguchi (1988), and Watanabe *et al.* (1983).

In the following we study the generalized Lagrange spaces M^m with the metric (I), we find the canonical nonlinear connection determined by the metric $g_{ij}(x, y)$, and we prove Synge's theorem (Theorem 3.1). Then we determine the canonical metrical d-connection of M^m, and its curvatures and torsions. The h- and v-electromagnetic tensors F_{ij} and f_{ij} are determined by means of the deflection tensors of the space. Theorem 5.2 gives us the Maxwell equations for F_{ij} and f_{ij}. For the nondispersive media (in which $\partial n/\partial y^i = 0$), the v-electromagnetic tensor $f_{ij}(x, y)$ vanishes and we have a simpler form (5.8) for the Maxwell equations.

If we consider the canonical metrical d-connection $L\Gamma(N) = (L^i_{jk}, C^i_{jk})$ as the deformation (6.1) of the d-connection $L\mathring{\Gamma}(N) = (\{^{\,i}_{jk}\}, C^i_{jk})$, we can express the geometrical objects of $L\Gamma(N)$ as functions of the corresponding geometrical objects of $L\mathring{\Gamma}(N)$. This idea allows us to give the explicit Einstein equations of the generalized Lagrange spaces endowed with the metric (I) (Section 7). Finally, in Section 8, we display the almost Hermitian model H^{2m} of the generalized Lagrange space M^m, which shows that the methods of symplectic geometry used in the theoretical mechanics cannot be applied for the study of the geometrical theory of the generalized Lagrange spaces M^m endowed with the metric (of the Synge type) (I).

The restriction of this theory to the cross section $S_V(M)$ gives us a theory of gravitation and electromagnetism for dispersive media with refractive index $n(x, V(x))$ endowed with the Synge metric (b). When $n(x, V(x)) = 1$ we have the classical Einstein equations and when $1/n^2 = 1 - 1/c^2$ we have the theory of gravitation and electromagnetism for spaces with the Beil metric (a′) (Kawaguchi and Miron, 1989*a*,*b*).

1. THE SYNGE METRIC

Let M be a $(C^\infty - m)$-dimensional real manifold (in particular $m = 4$), $\pi: TM \to M$ the tangent bundle of M, and (x^i, y^i) $(i, j, k, \ldots = 1, \ldots, m)$ the local coordinates on the total space TM. A transformation of coordinates $(x, y) \to (\bar{x}, \bar{y})$ has the form

$$\bar{x}^i = \bar{x}^i(x^1, \ldots, x^m), \qquad \operatorname{rank} \left\| \frac{\partial \bar{x}^i}{\partial x^j} \right\| = m$$
$$\bar{y}^i = \frac{\partial \bar{x}^i}{\partial x^j} y^j \tag{1.1}$$

Suppose that $\gamma_{ij}(x)$, $x \in M$, is a pseudo-Riemannian metric on the base manifold M. Then for a point $\tilde{u} \in TM$, with $\pi(\tilde{u}) = x$, $\gamma_{ij}(\pi(\tilde{u}))$ gives us a d-tensor field on TM, symmetric, covariant of order two, and of rank m.

Also, $y^i\, \partial/\partial x^i$ is a vector field on TM, called the *Liouville vector field*. Therefore

$$y_i = \gamma_{ij}(x) y^j \tag{1.2}$$

is a d-covector field on TM.

We denote

$$\|y\|^2=\gamma_{ij}(x)y^iy^j \tag{1.3}$$

and consider the differentiable manifold $\widetilde{TM}=TM\setminus\{0\}$, where $\{0\}$ is the null section of the projection $\pi: \widetilde{TM}\to M$.

Consequently, $\|y\|^2\neq 0$ on TM.

Assume that there is given a positive function $n(x,y)$ on $\widetilde{TM}$ and we take

$$u(x,y)=\frac{1}{n(x,y)} \tag{1.4}$$

The function $n(x,y)$ is called the *refractive index*.

We denote

$$a(x,y)=1+\left[1-\frac{1}{n^2(x,y)}\right]\|y\|^2 \tag{1.4'}$$

Now we consider

$$g_{ij}(x,y)=\gamma_{ij}(x)+[1-u^2(x,y)]y_iy_j \tag{1.5}$$

We have the following result.

Theorem 1.1. The following properties hold:

1. $g_{ij}(x,y)$ is a d-tensor field on $\widetilde{TM}$, covariant of order two, and symmetric.
2. $\operatorname{rank}\|g_{ij}(x,y)\|=m$

Proof. The first part of the theorem is immediate. For the second part, let us consider the d-tensor field

$$g^{ij}(x,y)=\gamma^{ij}(x)-\frac{1}{a}(1-u^2)y^iy^j \tag{1.6}$$

It is easy to check that

$$g_{ij}(x,y)g^{jk}(x,y)=\delta_i^k \tag{1.7}$$

and the theorem is proved. ■

Obviously, the refractive index $n(x,y)$ enters in the expression of the d-tensor field $g_{ij}(x,y)$.

Corollary 1.1. The pair $M^m=(M,g_{ij}(x,y))$ is a generalized Lagrange space.

$g_{ij}(x,y)$ is called the *fundamental tensor* or *metric tensor* of the generalized Lagrange space M^m.

Corollary 1.2:

1. $n(x, y) = 1$ implies that M^m coincides with the Riemannian space $V^m = (M, \gamma_{ij}(x))$.

2. $1/n^2 = 1 - 1/c^2$ implies that $g_{ij}(x, y)$ is reduced to the metric $\gamma_{ij}(x) + (1/c^2) y_i y_j$.

Remark. The metric from the Corollary 1.2, part 2 is considered in Kawaguchi and Miron (1988*a*,*b*) and Miron and Kawaguchi (1991*a*,*b*) and was suggested by the Beil metric (a′). We apply in the study of the generalized Lagrange spaces M^m endowed with the metric (1.5) the same methods used in the above papers.

We assume that on the manifold M there is a C^∞ nonnull vector field $V^i(x)$, $x \in M$. In this case, we have the following result.

Proposition 1.1. The mapping $S_V: M \to TM$, given by

$$x^i = x^i, \qquad y^i = V^i(x), \qquad x \in M \qquad (i = 1, \ldots, m) \tag{1.8}$$

is a cross section of the projection $\pi: TM \to M$.

Therefore $S_V(M)$ is an immersed submanifold in TM. It is called the section $S_V(M)$.

The restriction to the section $S_V(M)$ of the fundamental tensor $g_{ij}(x, g_{ij}(x, y))$ of the generalized Lagrange space M^m is the tensor field $g_{ij}(x, V(x))$ given by

$$\begin{aligned} g_{ij}(x, V(x)) &= \gamma_{ij}(x) + \left(1 - \frac{1}{n^2(x, V(x))}\right) V_i V_j \\ V_i(x) &= \gamma_{ij}(x) V^j(x) \end{aligned} \tag{1.9}$$

This is just the metric given by Synge (1966) and used in the study of the propagation of electromagnetic waves in a medium with index of refraction $n(x, V(x))$, $V^i(x)$ being the velocity field of the medium.

Definition 1.1. The medium $\mathscr{M} = [M, V(x), n(x, V(x))]$ is called a dispersive medium.

If $\partial n / \partial y^i = 0$, then $\mathscr{M}$ is called a nondispersive medium.

Definition 1.2. The restriction of the generalized Lagrange space M^m to the section $S_V(M)$ is called the geometrical model of the dispersive medium $\mathscr{M}$ endowed with the Synge metric (1.9).

For this reason the geometrical theory of the generalized Lagrange space M^m is considered by us as the relativistic geometrical optics of the medium $\mathscr{M}$. Therefore, we study geometrical properties of the space M^m

and consider the restriction to the section $S_V(M)$ in order to obtain the geometrical properties of the medium $\mathscr{M}$.

2. v-CANONICAL METRICAL d-CONNECTION

The vertical part of a d-connection, metrical with respect to the tensor metric $g_{ij}(x, y)$, is given by the d-tensor field

$$C^i_{jk} = \tfrac{1}{2} g^{ih}(\dot{\partial}_j g_{hk} - \dot{\partial}_k g_{jh} - \dot{\partial}_h g_{jk}) \tag{2.1}$$

where we set $\dot{\partial}_i = \partial/\partial y^i$.

By means of (1.5) we have

$$\dot{\partial}_k g_{ij} = [(1-u^2)(\gamma_{ik} y_j + \gamma_{jk} y_i) - 2u \dot{\partial}_k u y_i y_j] \tag{2.2}$$

Then, putting

$$\begin{aligned} \overset{\circ}{C}_{ijk} &= (1-u^2)\gamma_{ik} y_j, \qquad \overset{\circ}{C}{}^i_{jk} = g^{ih} \overset{\circ}{C}_{jhk} \\ \overset{1}{C}_{ijk} &= -u(y_i y_j \dot{\partial}_k u + y_j y_k \dot{\partial}_i u - y_k y_i \dot{\partial}_j u) \end{aligned} \tag{2.3}$$

we get the following result.

Theorem 2.1. The v-canonical metrical d-connection $C^i_{jk}(x, y)$ is given by

$$C^i_{jk} = \overset{\circ}{C}{}^i_{jk} + \overset{1}{C}{}^i_{jk} \tag{2.4}$$

where

$$\overset{\circ}{C}{}^i_{jk} = \frac{1}{a}(1-u^2)\gamma_{jk} y^i, \qquad \overset{1}{C}{}^i_{jk} = g^{ih} \overset{1}{C}_{jhk} \tag{2.4'}$$

Proof. A straightforward calculation leads to

$$g_{jh} C^h_{ik} = \overset{\circ}{C}_{ijk} + \overset{1}{C}_{ijk}$$

However, we have

$$g^{ij} y_j = \frac{1}{a} y^i, \qquad g_{ij} y^j = a y_i \tag{2.5}$$

Then (2.4) and (2.4′) hold. ■

Proposition 2.1. The medium $\mathscr{M}$ is nondispersive if and only if the d-tensor field $\overset{1}{C}{}^i_{jk}$ vanishes.

The vertical part C^i_{jk} of a d-connection allows us to consider the v-covariant derivative of the d-tensor fields. For example, in the case of the d-tensor field $K^i_j(x, y)$ the v-covariant derivative is

$$K^i_j|_h = \dot{\partial}_h K^i_j + C^i_{rh} K^r_j - C^r_{jh} K^i_r$$

As an application we note

$$\begin{aligned} g_{ij|k} &= 0, \qquad \|y\|^2|_k = 2y_k \\ y^i|_k &= d^i{}_k = \delta^i_k + y^h C^i_{hk} \end{aligned} \tag{2.6}$$

Here $d^i{}_k$ is the *v-deflection* tensor of the generalized Lagrange space M^m.

Also we get

$$a|_k = 2[(1-u^2)y_k - u\|y\|^2 \dot{\partial}_k u] \tag{2.6'}$$

Definition 2.1. The generalized Lagrange space M^m is called reducible to a Lagrange space if there exists a function $L\colon TM \to R$, of the class C^∞ on TM, continuous on the null section, such that

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \, \partial y^j} \tag{*}$$

It is interesting to note the following result.

Theorem 2.2. The generalized Lagrange space M^m with the metric (1.5) and $n(x, y) \neq 1$ is not reduced to a Lagrange space.

Proof. Let us suppose that there exists a Lagrangian $L\colon TM \to R$ which is a solution of the equation with partial derivatives (*). It follows that the d-tensor field $\dot{\partial}_k g_{ij}$ from (2.2) is totally symmetric. But the equations $\dot{\partial}_k g_{ij} - \dot{\partial}_i g_{kj} = 0$ imply

$$(1-u^2)(\gamma_{jk} y_i - \gamma_{ji} y_k) - 2u y_j (y_i \dot{\partial}_k u - y_k \dot{\partial}_i u) = 0$$

Contracting by y^j, we obtain $y_i \dot{\partial}_k u - y_k \dot{\partial}_i u = 0$ and $\gamma_{jk} y_i - \gamma_{ji} y_k = 0$. A new contraction by γ^{jk} in the last equation gives $(m-1)y_i = 0$ or $y_i = 0$ on TM. This is a contradiction. ■

3. THE NONLINEAR CONNECTION DETERMINED BY g_{ij}

The fundamental tensor field $g_{ij}(x, y)$ from (1.5) of the generalized Lagrange space M^m is well determined by the pseudo-Riemannian metric $\gamma_{ij}(x)$ and the refractive index $n(x, y)$. Therefore we can assume the following:

Postulate. The functions

$$N^i_{\ j}=\left\{ {i \atop jk} \right\} y^k \tag{3.1}$$

are the coefficients of the canonical nonlinear connection of the generalized Lagrange space M^m.

Of course $\{ {i \atop jk} \}$ are the Christoffel symbols of the metric $\gamma_{ij}(x)$. The arguments which support this postulate are:

1. The horizontal geodesics of the nonlinear connection N with the coefficients (3.1) are given by

$$\frac{d^2x^i}{dt^2}+\left\{ {i \atop jk} \right\}\frac{dx^j}{dt}\frac{dx^k}{dt}=0 \tag{3.2}$$

2. We have the following very interesting result of Synge (1966).

Theorem 3.1. (J. L. Synge). For a nondispersive medium the extremals of the integral of action on a curve c: $[0, 1] \to M$,

$$I(c)=\int_0^1 \mathscr{E}\left(x, \frac{dx}{dt}\right) dt, \qquad \mathscr{E}(x, y)=g_{ij}(x, y)y^i y^j \tag{3.3}$$

which have the property

$$g_{ij}\left(x, \frac{dx}{dt}\right)\frac{dx^i}{dt}\frac{dx^j}{dt}=0 \tag{3.4}$$

are the geodesics of the pseudo-Riemannian space $V^m=(M, \gamma_{ij}(x))$.

Proof. The extremals of $I(c)$ are given by the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathscr{E}}{\partial y^i}\right)-\frac{\partial \mathscr{E}}{\partial x^i}=0, \qquad y^i=\frac{dx^i}{dt} \tag{3.5}$$

But, in the hypothesis (3.4) and by $\partial n/\partial y^i=0$ the system of equations (3.5) leads to (3.2). ■

3. In the case $1/n^2=1-1/c^2$ the canonical nonlinear connection N (Kawaguchi and Miron, 1989*a*,*b*) of M^m is just (3.1).

Now we put

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N^j_{\ i}\frac{\partial}{\partial y^j} \qquad \left(\text{or } \delta_i=\frac{\delta}{\delta x^i}, \quad \partial_i=\frac{\partial}{\partial x^i}\right) \tag{3.6}$$

We have $(\delta_i, \dot{\partial}_i)$ a local basis of the module of vector fields $\mathscr{X}(TM)$ adapted to the horizontal distribution N determined by the nonlinear connection (3.1) and to the vertical distribution V on TM.

Proposition 3.1. The Berwald connection determined by the nonlinear connection (3.1) has the coefficients $B\Gamma = (\{^{\,i}_{jk}\}y^k, \{^{\,i}_{jk}\}, 0)$.

Proposition 3.2. The horizontal curves $c: I \subset R \to TM$ are characterized by the differential equations

$$x^i = x^i(t), \qquad \frac{dy^i}{dt} + \left\{ {i \atop jk} \right\} y^j \frac{dx^k}{dt} = 0, \qquad t \in I$$

Proposition 3.3. The horizontal distribution N determined by the nonlinear connection (3.1) is integrable if and only if the Riemannian manifold V^m is flat.

4. THE CANONICAL METRICAL *d*-CONNECTION

Now we can determine the horizontal part of the canonical metrical *d*-connection, which depends only on the fundamental tensor field $g_{ij}(x, y)$ from (1.5). So we have some general results:

Theorem 4.1 (R. Miron). There exists a unique *d*-connection $L\Gamma = (N^i{}_j, L^i_{jk}, C^i_{jk})$ for which:

1. $N^i{}_j$ is given by (3.1).
2. $g_{ij|k} = 0$, $g_{ij}|_k = 0$.
3. The torsions $T^i{}_{jk}$ and $S^i{}_{jk}$ of $L\Gamma$ vanish.

Theorem 4.2. The *d*-connection $L\Gamma$ which has the properties 1–3 from the last theorem has the coefficients $N^i{}_j$, C^i_{jk} given by (3.1), (2.4), and (2.4′) and L^i_{jk} given by the "generalized Christoffel symbols":

$$L^i_{jk} = \tfrac{1}{2} g^{ih}(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}) \tag{4.1}$$

Clearly, the *d*-connection $L\Gamma$ has the coefficients $N^i{}_j$, D^i_{jk}, and C^i_{jk} constructed from the fundamental tensor $g_{ij}(x, y)$ alone. For this reason it is called the *canonical metrical d-connection* of the generalized Lagrange space M^m. We denote also $L\Gamma$ by $L\Gamma(N) = (L^i_{jk}, C^i_{jk})$, N being given by (3.1).

The coefficients $L^i_{jk}(x, y)$ from (4.1) give the horizontal part (*h*-part) of $L\Gamma(N)$.

Proposition 4.1. The coefficients of the h-part of the canonical metrical d-connection $L\Gamma(N)$ are symmetric and can be put in the following form:

$$L^i_{jk} = \begin{Bmatrix} i \\ jk \end{Bmatrix} + \Lambda^i_{jk} \tag{4.2}$$

where

$$\Lambda^i_{jk} = -ug^{ih}(y_h y_j \delta_k u + y_h y_k \delta_j u - y_j y_k \delta_h u) \tag{4.2'}$$

Proposition 4.2. For a nondispersive medium $\mathscr{M}$ the d-tensor field Λ^i_{jk} vanishes if and only if the refractive index $n(x)$ is constant.

Proof. The d-tensor Λ^i_{jk} vanishes if and only if $\delta_k u = \partial_k u = 0$. Then $n(x) = \text{const}$ and conversely.

The h-part of the canonical metrical d-connection $L\Gamma(N)$ allows one to construct an h-covariant derivative (Miron and Anastasiei, 1987). For example, in the case of a d-tensor field $K^i_j(x, y)$ we have the h-covariant derivative with respect to $L\Gamma(N)$:

$$K^i_{j|h} = \delta_h K^i_j + L^i_{rh} K^r_j - L^r_{jh} K^i_r \quad \blacksquare$$

Proposition 4.3. The Ricci identities for a d-tensor field $K^i_j(x, y)$ with respect to the canonical metrical d-connection $L\Gamma(N)$ are given by

$$\begin{aligned} &K^i_{j|h|k} - K^i_{j|k|h} = K^s_j R_s{}^i{}_{hk} - K^i_s R_j{}^s{}_{hk} - K^i_j|_s R^s{}_{hk} \\ &K^i_{j|h}|_k - K^i_j|_{k|h} = K^s_j P_s{}^i{}_{hk} - K^i_s P_j{}^s{}_{hk} - K^i_{j|s} C^s_{hk} - K^i_j|_s P^s{}_{hk} \\ &K^i_j|_h|_k - K^i_j|_k|_h = K^s_j S_s{}^i{}_{hk} - K^i_s S_j{}^s{}_{hk} \end{aligned} \tag{4.3}$$

where

$$R^i{}_{jk} = \delta_k N^i{}_j - \delta_j N^i{}_k, \qquad P^i{}_{jk} = \dot{\partial}_k N^i{}_j - L^i_{kj} \tag{4.4}$$

are the torsions and

$$\begin{aligned} &R_j{}^i{}_{hk} = \delta_k L^i_{jh} - \delta_h L^i_{jk} + L^r_{jh} L^i_{rk} - L^r_{jk} L^i_{rh} + C^i_{jr} R^r{}_{hk} \\ &P_j{}^i{}_{hk} = \dot{\partial}_k L^i_{jh} - C^i_{jk|h} + C^i_{jr} P^r{}_{hk} \\ &S_j{}^i{}_{hk} = \dot{\partial}_k C^i_{jh} - \dot{\partial}_h C^i_{jk} + C^r_{jh} C^i_{rk} - C^r_{jk} C^i_{rh} \end{aligned} \tag{4.5}$$

are the curvatures of $L\Gamma(N)$.

Applying the Ricci identities to the fundamental tensor $g_{ij}(x, y)$ from (1.5), taking into account $g_{ij|k} = 0$ and $g_{ij}|_k = 0$, and denoting as usual $R_{ijhk} = g_{jr} R_i{}^r{}_{hk}$, etc., we have the following result.

Proposition 4.4. The curvature tensors of the canonical metrical d-connection $L\Gamma(N)$ have the properties

$$R_{ijhk}+R_{jihk}=0, \qquad P_{ijhk}+P_{jihk}=0$$
$$S_{ijhk}+S_{jihk}=0 \tag{4.6}$$

Let us consider the h-deflection tensor field

$$D^i{}_j=y^i{}_{|j} \tag{4.7}$$

By means of h- and v-deflection tensors $D^i{}_j$ and $d^i{}_j$ we can consider their covariant forms D_{ij} and d_{ij} given by $D_{ij}=g_{ir}D^r{}_j$ and $d_{ij}=g_{ij}d^r{}_j$.

Applying the Ricci identities to the Liouville vector field y^i, we have the following result.

Proposition 4.5. The h- and v-covariant deflection tensors D_{ij} and d_{ij} of the canonical metrical d-connection $L\Gamma(N)$ satisfy the equations

$$D_{ij|k}-D_{ik|j}=R_{0ijk}R^r{}_{jk}$$
$$D_{ij}|_k-d_{ik|j}=P_{0ijk}-D_{ir}C^r_{jk}-d_{ir}P^r{}_{jk} \tag{4.8}$$
$$d_{ij}|_k-d_{ik}|_j=S_{0ijk}$$

where by the index "0" we denote the contraction by y^i, i.e., $R_{0ijk}=y^rR_{rijk}$, etc.

5. ELECTROMAGNETIC TENSORS

In the case when the medium $\mathscr{M}$ is nondispersive we find an electromagnetic tensor field given by the skew-symmetric part of the h-covariant deflection tensor D_{ij}. In this case the skew-symmetric part of the v-covariant deflection tensor d_{ij} vanishes.

Generally, when the considered medium $\mathscr{M}$ is dispersive two electromagnetic tensors appear in the generalized Lagrange space M^m.

Definition 5.1. We call the h- and v-electromagnetic tensor fields of the generalized Lagrange space M^m the d-tensors

$$F_{ij}=\tfrac{1}{2}(D_{ij}-D_{ji}), \qquad f_{ij}=\tfrac{1}{2}(d_{ij}-d_{ji}) \tag{5.1}$$

respectively.

We have the following result.

Theorem 5.1. The h- and v-electromagnetic tensors of the space M^m have the form

$$F_{ij} = u\|y\|^2 (y_j \delta_i u - y_i \delta_j u)$$
$$f_{ij} = u\|y\|^2 (y_j \dot{\partial}_i u - y_i \dot{\partial}_j u) \tag{5.2}$$

Proof. The formulas

$$D^i{}_j = L^i_{0j} - N^i{}_j = \Lambda^i_{0j}, \qquad D_{ij} = g_{ir} \Lambda^r_{0j} \tag{5.3}$$

$$d^i{}_j = \delta^i_j + C^i_{0j}, \qquad d_{ij} = g_{ij} + g_{ir} C^r_{0j} \tag{5.3'}$$

lead to (5.2). ■

Corollary 5.1. If $\mathscr{M}$ is a nondispersive medium, then the v-electromagnetic tensors field $f_{ij}(x, y)$ vanishes.

These two electromagnetic tensors $F_{ij}(x, y)$ and $f_{ij}(x, y)$ are related by fundamental equations given by the following theorem.

Theorem 5.2. The h- and v-electromagnetic tensors $F_{ij}(x, y)$ and $f_{ij}(x, y)$ satisfy the Maxwell equations

$$F_{ij|k} + F_{jk|i} + F_{ki|j} \tfrac{1}{2} \underset{ijk}{\mathfrak{S}} (R_{0ijk} - d_{ir} R^r{}_{jk})$$

$$F_{ij}|_k + F_{jk}|_i + F_{ki}|_j = -(f_{ij|k} + f_{jk|i} + f_{ki|j}) \tag{5.4}$$

$$f_{ij}|_k + f_{jk}|_i + f_{ki}|_j = 0$$

Proof. The first of equations (5.4) is obtained from equations (4.8) by a cyclic permutation of the indices i, j, k and summing up. The second of equations (5.4) is a consequence of the last equality (4.8), taking into account one of the Bianchi identities satisfied by the canonical metrical d-connection $L\Gamma(N)$: $\mathfrak{S}_{ijk} S_{hijk} = 0$. Now, by means of the relations $C^i_{jk} = C^i_{kj}$ and $P^i{}_{jk} = P^i{}_{kj}$ the second of equations (4.8) gives

$$(F_{ij}|_k + F_{jk}|_i + F_{ki}|_j) + (f_{ij|k} + f_{jk|i} + f_{ki|j})$$
$$= \tfrac{1}{2} \underset{ijk}{\mathfrak{S}} (P_{0ijk} - P_{0jik})$$

However, we have another Bianchi identity (Miron and Anastasiei, 1987):

$$P_j{}^h{}_{ik} - P_k{}^h{}_{ij} = P^h{}_{ik}|_j - P^h{}_{ij}|_k + P^h{}_{rk} C^r{}_{ij} - P^h{}_{rj} C^r{}_{ik}$$

which lead to $\mathfrak{S}_{ijk} (P_{0ijk} - P_{0jik}) = 0$. Therefore relations (5.4) hold. ■

It is convenient to give a new form to the second member of the first of equations (5.4).

Lemma 5.1. The canonical metrical d-connection $L\Gamma(N)$ satisfies the identity

$$\underset{ijk}{\mathfrak{S}} (R_{0ijk} - d_{ir}R^{r}{}_{jk}) = -\underset{ijk}{\mathfrak{S}} [(C_{i0r} + d_{ir})R^{r}{}_{jk}] \tag{5.5}$$

Proof. The Bianchi identity from $L\Gamma(N)$,

$$\underset{ijk}{\mathfrak{S}} R_{i}{}^{h}{}_{jk} = \underset{ijk}{\mathfrak{S}} C_{i}{}^{h}{}_{r} R_{jk}$$

gives

$$\underset{ijk}{\mathfrak{S}} R_{0ijk} = -\underset{ijk}{\mathfrak{S}} C_{i0r} R^{r}{}_{jk}$$

such that (5.5) is satisfied.

Denoting by $r_{j}{}^{i}{}_{hk}(x)$ the curvature tensor of the Levi-Civita connection $\left\{ {i \atop jk} \right\}$,

$$r_{j}{}^{i}{}_{hk} = \partial_{k}\left\{ {i \atop jh} \right\} - \partial_{h}\left\{ {i \atop jk} \right\} + \left\{ {s \atop jh} \right\}\left\{ {i \atop sk} \right\} - \left\{ {s \atop jk} \right\}\left\{ {i \atop sh} \right\} \tag{5.6}$$

we have the following result.

Lemma 5.2. The equality

$$R^{h}{}_{ij} = r_{0}{}^{h}{}_{ij} \tag{5.7}$$

holds.

Indeed, (4.4) and (3.1) imply (5.7).

Corollary 5.2. The first of equations (5.4) is equivalent to

$$F_{ij|k} + F_{jk|i} + F_{ki|j} = -\tfrac{1}{2} \underset{ijk}{\mathfrak{S}} [(C_{i0s} + d_{is}) r_{0}{}^{s}{}_{jk}] \tag{5.4$'$}$$

Now we can prove an important result:

Theorem 5.3. If the medium $\mathscr{M}$ is nondispersive, then the Maxwell equations of the generalized Lagrange space M^{m} have the form

$$\begin{aligned} F_{ij|k} + F_{jk|i} + F_{ki|j} &= 0 \\ F_{ij}|_{k} + F_{jk}|_{i} + F_{ki}|_{j} &= 0 \end{aligned} \tag{5.8}$$

Proof. For the nondispersive medium $\mathcal{M}$ the v-electromagnetic tensor field $f_{ij}(x, y)$ vanishes. Then the Maxwell equations (5.4), (5.4′) give

$$\underset{ijk}{\mathfrak{S}} F_{ij|k} = -\tfrac{1}{2} \underset{ijk}{\mathfrak{S}} [(C_{i0s} + d_{is}) r_0{}^s{}_{jk}]$$

$$= -\frac{1}{2}\left[a y^s \underset{ijk}{\mathfrak{S}} r_{sijk} + 2(1-u^2) y^p y^q \underset{ijk}{\mathfrak{S}} y_i r_{pqjk} \right] = 0$$

and

$$\underset{ijk}{\mathfrak{S}} F_{ij}|_k = 0 \quad \blacksquare$$

Remarks. There exist other particularly interesting cases:

1. The Riemannian space V^m is flat.
2. The generalized Lagrange space M^m is a locally Minkowski space.

6. REMARKABLE TRANSFORMATION OF CONNECTIONS

The direct study of the canonical metrical d-connection $L\Gamma(N)$ is very difficult. Considering $L\Gamma(N)$ as a deformation of the d-connection $L\mathring{\Gamma}(N) = (\{{}^{i}_{jk}\}, \mathring{C}^i_{jk})$, with the tensors of deformation $(\Lambda^i_{jk}, \overset{1}{C}{}^i_{jk})$, we have the transformation of d-connections $L\mathring{\Gamma}(N) \to L\Gamma(N)$ given by

$$L^i_{jk} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} + \Lambda^i_{jk}, \qquad C^i_{jk} = \mathring{C}^i_{jk} + \overset{1}{C}{}^i_{jk} \tag{6.1}$$

We will study the effect of previous transformations of the curvature tensors of $L\Gamma(N)$. In this respect, denoting with a diacritic "∘" the geometrical objects determined by the d-connection $L\Gamma(N)$, we can prove the following result.

Proposition 6.1. The d-connection $L\mathring{\Gamma}(N)$ has the torsions

$$\mathring{T}^i{}_{jk} = \mathring{S}^i{}_{jk} = 0, \qquad \mathring{R}^i_{jk} = r_0{}^i{}_{jk}, \qquad \mathring{C}^i_{jk}, \mathring{P}^i{}_{jk} = 0 \tag{6.2}$$

Proposition 6.2. The curvatures of the d-connection $L\mathring{\Gamma}(N)$ are given by

$$\begin{gathered} \mathring{R}_j{}^i{}_{kh} = g^{is} r_{jskh}, \qquad \mathring{P}_j{}^i{}_{kh} = -\mathring{C}^i_{jh\mathring{|}k} \\ \mathring{S}_j{}^i{}_{kh} = \frac{1}{a}(i-u^2) g^{is}\left[(\gamma_{jk}\gamma_{sh} - \gamma_{jh}\gamma_{sk}) - \frac{2u}{a^2}(\gamma_{jk}\dot{\partial}_h u - \gamma_{jh}\dot{\partial}_k u) \right] \end{gathered} \tag{6.3}$$

where $\mathring{|}$ is the h-covariant derivative with respect to $L\mathring{\Gamma}(N)$.

For nondispersive media $\mathscr{M}$ the v-curvature tensor $\mathring{S}_{j\ kh}^{\ i}$ has a very simple form,

$$\mathring{S}_{j\ kh}^{\ i}=\frac{1}{a}(1-u^2)g^{is}(\gamma_{jk}\gamma_{sh}-\gamma_{jh}\gamma_{sk})$$

Proposition 6.3. The deflection tensors $\mathring{D}^i_{\ j}$ and $\mathring{d}^i_{\ j}$ of the d-connection $L\mathring{\Gamma}(N)$ satisfy the equations

$$\mathring{D}^i_{\ j}=0, \qquad \mathring{d}^i_{\ j}=\delta^i_j+\frac{1}{a}(1-u^2)y^iy_j \tag{6.4}$$

Proposition 6.4. The h- and v-electromagnetic tensors $\mathring{F}_{ij}$ and $\mathring{f}_{ij}$ of the d-connection $L\mathring{\Gamma}(N)$ vanish.

Now we can prove an important result:

Theorem 6.1. The tensors of deformation Λ^i_{jk}, $\overset{1}{C}{}^i_{jk}$ have the forms, respectively,

$$\begin{aligned}\Lambda^i_{jk}&=\tfrac{1}{2}g^{is}(g_{js\mathring{|}k}+g_{sk\mathring{|}j}-g_{jk\mathring{|}s})\\ \overset{1}{C}{}^i_{jk}&=\tfrac{1}{2}g^{is}(g_{js}\overset{\circ}{|}_k+g_{sk}\overset{\circ}{|}_j-g_{jk}\overset{\circ}{|}_s)\end{aligned} \tag{6.5}$$

where $\mathring{|}$ and $\overset{\circ}{|}$ mean the h- and v-covariant derivatives with respect to $L\mathring{\Gamma}(N)$, respectively.

Proof. From (4.2′) and (2.4′), taking into account

$$g_{ij\mathring{|}k}=-2uy_iy_j\delta_ku, \qquad g_{ij}\overset{\circ}{|}_k=-2uy_iy_j\dot{\partial}_ku \tag{6.6}$$

we get (6.5). ■

The torsions and curvatures of the canonical metrical d-connections $L\Gamma(N)$ can be computed by using the transformation of d-connections (6.1).

Proposition 6.5. The torsions of $L\Gamma(N)$ are

$$T^i_{\ jk}=S^i_{\ jk}=0, \qquad R^i_{\ jk}=r_0{}^i_{\ jk}, \qquad C^i_{jk},\ P^i_{\ jk}=-\Lambda^i_{jk} \tag{6.7}$$

Proposition 6.6. The curvature tensors of the canonical metrical d-connection $L\Gamma(N)$ are given by

$$\begin{aligned}R_{j\ kh}^{\ i}&=\mathring{R}_{j\ kh}^{\ i}+\rho_{j\ kh}^{\ i}\\ P_{j\ kh}^{\ i}&=\mathring{P}_{j\ kh}^{\ i}+\pi_{j\ kh}^{\ i}\\ S_{j\ kh}^{\ i}&=\mathring{S}_{j\ kh}^{\ i}+\sigma_{j\ kh}^{\ i}\end{aligned} \tag{6.8}$$

where we put

$$
\begin{aligned}
\rho_j{}^i{}_{kh} &= \Lambda^i_{jk\overset{\circ}{|}h} - \Lambda^i_{jh\overset{\circ}{|}k} + \Lambda^s_{jk}\Lambda^i_{sh} - \Lambda^s_{jh}\Lambda^i_{sk} \\
\pi^i_{jkh} &= -\overset{1}{C}{}^i_{jh\overset{\circ}{|}k} + \Lambda^i_{jk}\overset{\circ}{|}_h + \Lambda^s_{jk}\overset{1}{C}{}^i_{sh} - \overset{1}{C}{}^s_{jh}\Lambda^i_{js} + \Lambda^i_{js}\overset{\circ}{C}{}^s_{kh} \qquad (6.9) \\
\mathfrak{S}_j{}^i{}_{kh} &= \overset{1}{C}{}^i_{jk}\overset{\circ}{|}_h - \overset{1}{C}{}^i_{jh}\overset{\circ}{|}_k + \overset{1}{C}{}^s_{jk}\overset{1}{C}{}^i_{sh} - \overset{1}{C}{}^s_{jh}\overset{1}{C}{}^i_{sk}
\end{aligned}
$$

Evidently, for nondispersive media $\mathscr{M}$ we have the following result.

Proposition 6.7. If $\mathscr{M}$ is a nondispersive medium, the curvature tensors of $L\Gamma(N)$ are given by (6.8), in which

$$
\begin{aligned}
&\rho_j{}^i{}_{kh} = \Lambda^i_{jk\overset{\circ}{|}h} - \Lambda^i_{jh\overset{\circ}{|}k} + \Lambda^s_{jk}\Lambda^i_{sh} - \Lambda^s_{jh}\Lambda^i_{sk} \\
&\pi_j{}^i{}_{kh} = \Lambda^i_{jk\overset{\circ}{|}h} + \Lambda^i_{js}\overset{\circ}{C}{}^s_{kh}, \qquad \sigma_j{}^i{}_{kh} = 0
\end{aligned} \qquad (6.10)
$$

7. EINSTEIN EQUATIONS OF M^m

The Einstein equations of the generalized Lagrange space M^m endowed with the canonical metrical d-connection $L\Gamma(N)$ (Miron, 1985; Miron and Anastasiei, 1987; Miron *et al.*, 1991), restricted to a section $S_V(N)$, give us the Einstein equations of a medium $\mathscr{M}$ endowed with the Synge metric (1.9).

The following theorem is known (Miron and Anastasiei, 1987; Miron *et al.*, 1991).

Theorem 7.1. The Einstein equations of the generalized Lagrange space M^m endowed with the canonical metrical d-connection $L\Gamma(N)$ are given by

$$
\begin{aligned}
R_{ij} - \tfrac{1}{2}Rg_{ij} &= \kappa \overset{H}{T}_{ij}, \qquad \overset{1}{P}_{ij} = \kappa \overset{1}{T}_{ij} \\
S_{ij} - \tfrac{1}{2}Sg_{ij} &= \kappa \overset{V}{T}_{ij}, \qquad \overset{2}{P}_{ij} = -\kappa \overset{2}{T}_{ij}
\end{aligned} \qquad (7.1)
$$

where κ is a constant and

$$
\begin{aligned}
&R_{ij} = R_i{}^s{}_{js}, \qquad S_{ij} = S_i{}^s{}_{js}, \qquad \overset{1}{P}_{ij} = P_i{}^s{}_{js} \\
&\overset{2}{P}_{ij} = P_i{}^s{}_{sj}, \qquad R = g^{ij}R_{ij}, \qquad S = g^{ij}S_{ij}
\end{aligned} \qquad (7.2)
$$

Theorem 7.2. The divergences of the h- and v-energy-momentum $\overset{H}{T}_{ij}$ and $\overset{V}{T}_{ij}$ are given by

$$
\overset{H}{T}{}^i_{j|i} = -\tfrac{1}{2}(P^{ih}{}_{jr}R^r{}_{hi} + 2\overset{2}{P}{}^i{}_rR^r{}_{ij}), \qquad \overset{V}{T}{}^i{}_j|_i = 0 \qquad (7.3)
$$

Corollary 7.1. If the Riemannian space V^m is locally flat, then the following laws of conservation hold:

$$\overset{H}{T}{}^i{}_{j|i}=0, \qquad \overset{V}{T}{}^i{}_{j}|_i=0 \tag{7.4}$$

Corollary 7.2. If the generalized Lagrange space M^m has the property $P_j{}^i{}_{kh}=0$, then the laws of conservation (7.4) are satisfied.

Now we can express the Einstein tensors

$$\overset{H}{E}_{ij}=R_{ij}-\tfrac{1}{2}Rg_{ij}, \qquad \overset{V}{E}_{ij}=S_{ij}-\tfrac{1}{2}Sg_{ij} \tag{7.5}$$

by means of the Ricci tensors of the d-connection $L\Gamma(N)$. We have

$$\begin{aligned} R_{ij}&=\mathring{R}_{ij}+\rho_{ij}, \qquad & \overset{1}{P}_{ij}&=P_{ij}+\overset{1}{\pi}_{ij} \\ S_{ij}&=\mathring{S}_{ij}+\sigma_{ij}, \qquad & \overset{2}{P}_{ij}&=P_{ij}+\overset{2}{\pi}_{ij} \end{aligned} \tag{7.6}$$

where

$$\begin{aligned} \rho_{ij}&=\rho_i{}^s{}_{js}, \qquad & \sigma_{ij}&=\sigma_i{}^s{}_{js} \\ \overset{1}{\pi}_{ij}&=\pi_i{}^s{}_{js}, \qquad & \overset{2}{\pi}_{ij}&=\pi_i{}^s{}_{sj} \\ P_{ij}&=\mathring{P}_i{}^s{}_{js}, \qquad & P_{ij}&=\mathring{P}_i{}^s{}_{sj} \end{aligned} \tag{7.7}$$

Also we shall put

$$\rho=g^{ij}\rho_{ij}, \qquad \sigma=g^{ij}\sigma_{ij} \tag{7.8}$$

Now we can formulate the following theorem.

Theorem 7.3. The Einstein equations of the generalized Lagrange space M^m endowed with the canonical metrical d-connection $L\Gamma(N)$ are given by

$$\begin{aligned} \mathring{R}_{ij}+\rho_{ij}-\tfrac{1}{2}(R+\rho)g_{ij}&=\kappa\overset{H}{T}_{ij}, \qquad & P_{ij}+\overset{1}{\pi}_{ij}&=\kappa\overset{1}{T}_{ij} \\ \mathring{S}_{ij}+\sigma_{ij}-\tfrac{1}{2}(S+\sigma)g_{ij}&=\kappa\overset{V}{T}_{ij}, \qquad & P_{ij}+\overset{2}{\pi}_{ij}&=-\kappa\overset{2}{T}_{ij} \end{aligned} \tag{7.9}$$

The equations (7.9) give the change of the Einstein equations of the connection $L\mathring{\Gamma}(N)=(\{{}^{\,i}_{jk}\}, \mathring{C}^i_{jk})$ with respect to the transformation of the connection (6.1).

Now it is easy to describe the equations (7.9) by making use of (6.3), (6.8), and (6.9).

Also we can give particulars of all these equations in the case of a nondispersive medium.

We note also the following theorems.

Theorem 7.4. The h-paths of the canonical metrical d-connection $L\Gamma(N)$ are given by the system of differential equations

$$\frac{d^2x^i}{dt^2}+\left\{\begin{matrix} i \\ jk \end{matrix}\right\}\frac{dx^j}{dt}\frac{dx^k}{dt}=-\Lambda^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}$$

Theorem 7.5. The v-paths of the canonical metrical d-connection $L\Gamma(N)$ at a point $x_0 \in M$ are characterized by

$$\frac{d^2y^i}{dt^2}+\frac{1}{a(x_0, y)}\gamma_{jk}(x_0)y^i\frac{dy^j}{dt}\frac{dy^k}{dt}=-\overset{1}{C}{}^i_{jk}(x_0, y)\frac{dy^j}{dt}\frac{dy^k}{dt}$$

8. ALMOST HERMITIAN MODEL OF THE SPACE M^m

All the previous constructions have a good meaning on the so-called almost Hermitian model (Miron and Anastasiei, 1987).

We consider the generalized Lagrange space $M^m=(M, g_{ij}(x, y))$, where the fundamental tensor $g_{ij}(x, y)$ is given by (1.5). Taking into account the nonlinear connection N with the coefficients (3.1), as well as the fact that it determines a subbundle HTM of the tangent bundle TTM, we have the Whitney sum: $TTM=HTM\oplus VTM$, where VTM is the vertical subbundle of TTM.

In every point $\tilde{u}\in TM$ we have the fiber $N_{\tilde{u}}$ of HTM and the fiber $V_{\tilde{u}}$ of VTM. The tangent space $T_{\tilde{u}}TM$ is the direct sum of the vector spaces $N_{\tilde{u}}$ and $V_{\tilde{u}}$. Then the mappings

$$N:\quad \tilde{u}\to N_{\tilde{u}},\qquad V:\quad \tilde{u}\to V_{\tilde{u}}$$

are two supplementary distributions of TM called horizontal and vertical, respectively.

We consider the local adapted basis $(\delta/\delta x^i, \partial/\partial y^i)=(\delta_i, \dot{\partial}_i)$ to these two distributions N and V. This is a local basis of the module of the vector fields $\mathscr{X}(TM)$. The natural almost complex structure

$$\mathsf{F}:\quad \mathscr{X}(TM)\to\mathscr{X}(TM)$$

can be given on the local generators of $\mathscr{X}(TM)$ by

$$\mathsf{F}(\delta_i)=-\dot{\partial}_i,\qquad \mathsf{F}(\dot{\partial}_i)=\delta_i\qquad (i=1,\ldots,m) \tag{8.1}$$

Proposition 8.1. The almost complex structure F is integrable if and only if the Riemann space V^m is flat.

The dual basis of $(\delta_i, \dot{\partial}_i)$ is $(dx^i, \delta y^i)$, where

$$\delta y^i = dy^i + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} y^j \, dx^k \tag{8.2}$$

Now we can consider the N-lift of the fundamental tensor g_{ij}:

$$g_{ij}(x, y) = \gamma_{ij}(x) + \left(1 - \frac{1}{n^2(x, y)}\right) y_i y_j \tag{8.3}$$

that is,

$$G = g_{ij}(x, y) \, dx^i \otimes gx^j + g_{ij}(x, y) \, \delta y^i \otimes \delta y^j \tag{8.4}$$

Obviously, G is a symmetric, nondegenerate, covariant of order two tensor field, globally defined on the manifold $\widetilde{TM}$.

Theorem 8.1. The pair (G, F) in (8.1), (8.2) is an almost Hermitian structure on $\widetilde{TM}$.

Indeed, it is easy to prove that

$$G(\mathsf{F}X, \mathsf{F}Y) = G(X, Y), \qquad \forall X, Y \in \mathscr{X}(\widetilde{TM})$$

Therefore, the space $H^{2m} = (TM, G, \mathsf{F})$ is called the almost Hermitian model of the generalized Lagrange space M^m.

Assuming that on M there is a nonvanishing vector field $V^i(x)$, $x \in M$, and taking into account the cross section $S_V(M)$ given by (1.8), we can take the restriction to $S_V(M)$ of the space H^{2m}. This restriction gives us "the almost Hermitian model of the dispersive medium $\mathscr{M}$." Of course, the metric of $\mathscr{M}$ is the Synge metric (1.9).

Let us consider the 2-form of H^{2m}:

$$\theta(X, Y) = G(\mathsf{F}X, Y), \qquad \forall X, Y \in \mathscr{X}(TM) \tag{8.5}$$

We have the following result.

Proposition 8.2:

1. θ is an almost symplectic structure on $\widetilde{TM}$.
2. In the adapted basis $(\delta_i, \dot{\partial}_i)$, θ is expressed by

$$\theta = g_{ij}(x, y) \, \delta y^i \wedge dx^j \tag{8.5'}$$

When θ is a closed 2-form, H^{2m} is an almost Kählerian space.

We shall compute the exterior differential of the 2-form θ. To this aim, we note that the exterior differential of the 1-form δy^i is given by

$$d(\delta y^i) = -\tfrac{1}{2} R^i{}_{jk}\, dx^j \wedge dx^k - \frac{\partial N^i{}_j}{\partial y^k}\, dx^j \wedge \delta y^k$$

Then we obtain the exterior differential of the 2-form θ:

$$\begin{aligned} d\theta = & -\underset{ijk}{\mathfrak{S}}\, g_{ir} R^r{}_{jk}\, dx^i \wedge dx^j \wedge dx^k \\ & + \tfrac{1}{2}(-g_{ij} P^r{}_{jk} + g_{jr} P^r{}_{ik})\, dx^i \wedge dx^j \wedge \delta y^k \\ & + \tfrac{1}{2}(g_{jr} C^r_{ki} - g_{ir} C^r_{kj})\, \delta y^i \wedge \delta y^j \wedge dx^k \end{aligned}$$

A straightforward calculation leads to

$$\begin{aligned} d\theta = & \frac{1}{\|y\|^2} y_k F_{ij}\, dx^i \wedge dx^j \wedge \delta Y^K + \left[\frac{1}{2}(1-u^2)(\gamma_{ki} y_j - \gamma_{kj} y_i) \right. \\ & \left. - \frac{1}{\|y\|^2} y_k f_{ij}\right] \delta y^i \wedge \delta y^j \wedge dx^k \end{aligned} \tag{8.6}$$

We examine the case $d\theta = 0$. It implies

$$\begin{aligned} & y_k F_{ij} = 0 \\ & \tfrac{1}{2}(1-u^2)(\gamma_{ik} y_j - \gamma_{jk} y_i) - \frac{1}{\|y\|^2} y_k f_{ij} = 0 \end{aligned} \tag{8.7}$$

However, this system of equations is equivalent to

$$\begin{aligned} & F_{ij} = 0, \qquad f_{ij} = 0 \\ & \gamma_{ik} y_j - \gamma_{jk} y_i = 0 \end{aligned} \tag{8.8}$$

The equations (8.8) give a contradiction.
Consequently, we have the following theorem.

Theorem 8.2. The almost Hermitian model H^{2m} of the generalized Lagrange space M^m, endowed with the fundamental field (8.3), is not an almost Kählerian space.

This theorem is significant. It follows that we cannot apply the methods of symplectic geometry (Libermann and Marle, 1987) in order to study dispersive media $\mathcal{M}$ endowed with the Synge metric (1.9).

Now we can prove the following theorems:

Theorem 8.3. There is a unique linear connection ∇ on $\widetilde{TM}$ having the properties:

1. ∇ preserves by parallelism the vertical distribution V.
2. ∇ is an almost Hermitian connection:

$$\nabla_X G = 0, \qquad \nabla_X \mathsf{F} = 0, \qquad \forall X \in \mathscr{X}(\widetilde{TM})$$

3. The h- and v-torsions of ∇ vanish.

The connection ∇ from the previous theorem is called *canonical* for the almost Hermitian model H^{2m} of the generalized Lagrange space M^m.

Theorem 8.4. In the adapted basis $(\delta_i, \dot{\partial}^i)$ the canonical connection ∇ of the Hermitian model H^{2m} is determined by the coefficients (L^i_{jk}, C^i_{jk}) in (6.1) of the canonical metrical d-connection $L\Gamma(N)$ of the generalized Lagrange space M^m.

Theorem 8.5. The Einstein equations of the canonical connection of the almost Hermitian model H^{2m} are equivalent to the Einstein equations (7.9) of the generalized Lagrange space M^m endowed with the canonical metrical d-connection $L\Gamma(N)$.

9. CONCLUSIONS

We have studied the geometrical properties of a dispersive medium $\mathscr{M}$ with a refractive index $n(x, V(x))$ endowed with the Synge metric (I) from relativistic geometrical optics. In particular, we have paid attention to the case when $n(x, V(x))$ does not depend on the velocity $V(x)$ of the particles x of the considered medium $\mathscr{M}$. This is the nondispersive case.

We introduce the h- and v-electromagnetic tensors $F_{ij}(x, y)$ and $f_{ij}(x, y)$. They are derived only from the metric of the medium.

In the case of the nondispersive medium $\mathscr{M}$ the v-electromagnetic tensor $f_{ij}(x, y)$ vanishes and all our considerations become simple.

These two tensors F_{ij} and f_{ij} satisfy the remarkable Maxwell equations (5.4).

The Einstein equations (7.9) for the dispersive medium appear now for the first time; however, their expressions are rather complicated.

A good simplification is given for a nondispersive medium. It is important to note that the equations (7.9) can be explicitly written by considering the deformation (6.1) of the d-connections $L\mathring{\Gamma}(N)$ and $L\Gamma(N)$.

The laws of conservation for the energy-momentum tensors have also been studied.

There are a number of potential physical applications of this theory.

We also studied in the preceding section, the Hermitian model H^{2m} of the dispersive medium $\mathscr{M}$ and proved that H^{2m} is not reduced to an almost Kählerian space. This result is significant, because the associated almost symplectic structure θ of H^{2m} cannot be a symplectic structure. Consequently, *the geometrical properties from relativistic optics based on the Synge metric* (I) *cannot be studied by means of symplectic geometry as in theoretical mechanics*. This was already recognized by Synge (1966, p. 311).

In this respect, Ingarden (1987) made a very interesting remark: "Symplectic geometry is a geometry of mechanical phase space showing the sense of the canonical transformations of positions and momenta. However in mechanics without momenta there exist also potentials in the field theory of potential fields. Therefore we do not need the symplectic geometry but geometry of potentials and that is namely a metric geometry." Here we have confirmed these statements.

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